

Mathematical Modeling of Earthwork Optimization Problems

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Abstract

In the past this research efforts in optimizing earthwork processes focused mainly on minimizing transportation costs and mass haul distances, respectively. This kind of optimization problem, well known as *earthwork allocation problem* can be solved by applying linear programming techniques. As a result, the most cost-efficient cut-to-fill assignments will be found. In this article, starting from an optimal cut-to-fill assignment, we formulate a new corresponding combinatorial optimization problem. This *earthwork section division problem* arises when a large road project is divided into several linear construction sections and tendered to different normally non-cooperating construction companies. The optimization objective is to partition the optimized cut-to-fill-assignments in different earthwork sections with minimal earth movements between them. This problem is subjected to certain user-defined constraints, like number of sections, minimal and maximal section-length, etc. The proposed solution model will be integrated into an earthwork modeling and assessment system which allows performing a quantity take-off from a roadway model to provide the necessary input data for the optimization algorithms.

Keywords: earthwork optimization, linear programming, road construction, mathematical modeling.

1 Introduction

Earthwork is the major working task in road construction projects and characterized by large quantities of earth material which have to be excavated, transported, and filled, possibly over a long distance. Therefore, linear programming (LP) techniques have been applied in order to minimize the transportation costs and the mass haul distances in the earthwork processes, respectively. The first LP-model of this *earthwork allocation problem* has been formulated and developed by Stark and Mayer (1983), further studies and extensions of this model have been done by Easa (1987 and 1988), Jayawardane (1990) and Son (2005). As a result, the LP solution provides the optimal cut-fill-assignments and determines the corresponding amount of earth to be hauled.

Nowadays, large road construction projects, such as highway projects, will usually be tendered to different constructors or to sub-constructors by a general constructor. The divided sub-projects can be processed in parallel in order to reduce the overall project duration. Normally, it is difficult to establish some kind of cooperation between these constructors. Consequently, if the division of earthwork sections is not optimal, it may happen that one or more construction sections suffer from considerable overflow of earth materials while the other sections demand additional materials from external borrow pits. Although the material flows can be balanced during the construction phase with

a lot of coordination efforts, this will usually result in additional costs for the remitter. Accordingly, it is advantageous to solve the *earthwork section division problem* at a very early stage in order to support the tender or general constructor to make an optimal decision.

The optimization results of these two optimization problems can be integrated in existing computer-aided earthwork systems which have been developed in previous research efforts. This includes earthwork control systems (Askew et al., 2002, Kim et al., 2003), earthwork modeling and simulation systems (Chahrour 2007, Ji et al., 2009) and 4D virtual road construction frameworks (Söderström and Olofsson, 2007).

2 Earthwork allocation problem

In road construction projects, cut and fill areas are traditionally defined by intersecting the road level with the terrain level vertically (Figure 1a). The quantities of cut and fill areas can be calculated using numerical methods, depending on the national regulation in civil engineering, such as the Gauß-Elling-method applied in Germany (REB 1979). The mass haul distance can be defined as Euclidean distance between the centre points of cut and fill areas.

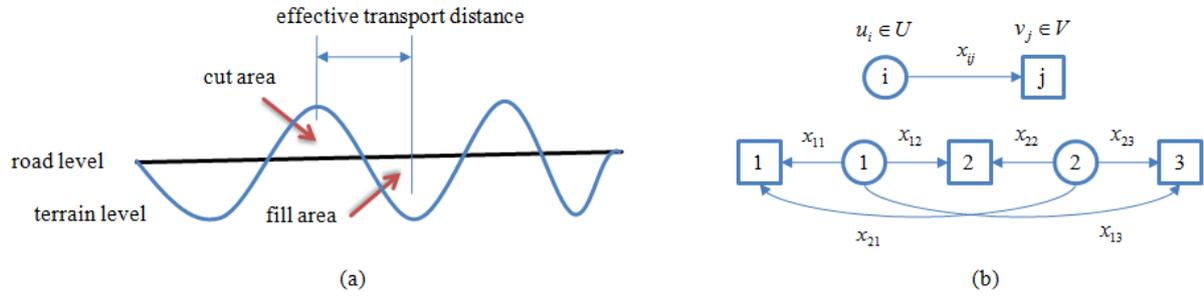


Figure 1. (a) Example of cut and fill areas in road construction project; (b) corresponding bipartite directed graph.

To formulate the optimization problem, we define $G = (P, E)$ to denote a bipartite graph which contains of a vertex set P and the edge set E . The set of vertices P is partitioned into two disjoint subsets U and V of P . The set U consists of those vertices corresponding to cut areas and, analogously, the set V represents vertices corresponding to fill areas. For each vertex $i \in P$, the parameter X_i denotes the amount of material to be sent (if $i \in U$) or to be filled (if $i \in V$). We may assume that the total amount to be sent equals the total amount to be filled by introducing dump sites and borrow pits: A dump site is used to dump earth material due to material overflow. A borrow pit provides filling materials which have been bought in addition. A directed edge e_{ij} is introduced for each pair of vertices (i, j) where i is a vertex corresponding to a cut area and j is a vertex corresponding to a fill area. Each of these edges mirrors the possibility of sending material from a cut area to a fill area. Additionally, each edge e_{ij} has an associated cost c_{ij} which represents the cost of transporting one mass unit of material from i to j .

A decision variable x_{ij} is assigned to each of the directed edges in the set E . It denotes the quantities of earth to be hauled from cut i to fill j following the edge direction (Figure 1b). We can model the *earthwork allocation problem* as a linear programming problem (cf. Figure 2). We assume that the (known) transportation cost along each edge (i, j) is non-negative, i.e., $c_{ij} \geq 0$. The objective function (1) is to minimize the total transportation cost. Due to the fact that in the real world only positive material flows make sense, the decision variables x_{ij} are restricted to be non-negative (see Constraint (4)). Constraint (2) implies that the total quantity of material to be hauled from some cut area i to all fill areas equals the total quantity of material X_i provided by cut i . Constraint (3) is similar to (2) for the requirements in j .

$$\min \sum_{(i,j) \in E} c_{ij} \cdot x_{ij} \quad (1)$$

$$\sum_{(i,j) \in E} x_{ij} = X_i \quad \text{for all } i \in U \quad (2)$$

$$\sum_{(i,j) \in E} x_{ij} = X_j \quad \text{for all } j \in V \quad (3)$$

$$x_{ij} \geq 0 \quad (4)$$

Figure 2. Mathematical formulation of earthwork allocation problem.

This formulation is a simplified *minimal cost flow problem* and can be solved efficiently using network flow algorithms (see Ahuja, 1993). Having solved the optimization model above, the amount of earth x_{ij} to be moved from a cut area i to a fill area j , such that the overall transportation cost is minimal, is known (Figure 1b). A real-world example will be presented in Section 4.

3 Earthwork section division problem

The *earthwork section division problem* emerges when a large road construction project is divided into several separate earthwork sections. The objective of this optimization problem is to obtain a reasonable division of the project such that in each earthwork section the quantities of excavated material and filling material are preferably balanced in order to avoid interactions between the sections. As mentioned before, we propose a two-step optimization algorithm for this problem. At first we solve the *earthwork allocation problem* in order to find a minimal cost cut-to-fill assignment. In the second step we identify the section division having the least necessary overall earth movement between different earthwork sections among all section divisions meeting the demands (such as desired number of earth sections or maximal length of a section).

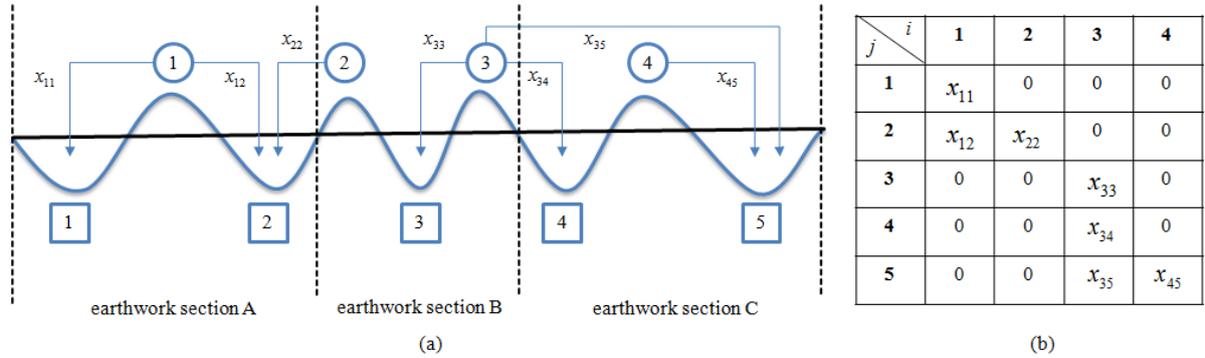


Figure 3. (a) Example of a feasible section division with three resulting earthwork sections $A = \{p_1 = v_1, p_2 = u_1, p_3 = v_2\}$, $B = \{p_4 = u_2, p_5 = v_3, p_6 = u_3\}$ and $C = \{p_7 = v_4, p_8 = u_4, p_9 = v_5\}$. (b) Cut-to-fill assignment matrix $(x_{ij})_{ij}$ obtained from solving the *earthwork allocation problem* optimally.

In order to be able to formulate this problem, we consider the set $P = U \cup V$ together with the index set $I = \{1, \dots, n\}$, representing the possible *positions* for a section division, i.e., the cut and fill areas ordered according to their actual appearance along the construction project. Hence, an earthwork section ES from position p_i to position p_j consists of all cuts and fills located in between: $ES = \{p_k \in P : i \leq k \leq j\}$. The required *material flow* between position p_i and position p_j is exactly the value x_{kl} obtained from the earthwork allocation problem, given that $p_i = u_k$ is a cut and $p_j = v_l$ is a fill, and zero otherwise. An example of a section division is illustrated in Figure 3.

The *earthwork section division problem* can be formulated as to find a feasible division of the project into earthwork sections, such that the total material flow between different sections is minimal. This combinatorial optimization problem can be expressed by a binary linear program (BP) with decision variables b_{ij} , representing the earthwork sections, which are interpreted as follows:

$$b_{ij} = \begin{cases} 1, & \text{if an earthwork section begins at position } i \text{ and ends at position } j \\ 0, & \text{else} \end{cases}$$

Obviously, not all possible combinations of those variables correspond to a feasible earthwork section division, e.g. it is not allowed to have gaps between the sections, and different sections must not overlap. Therefore, we need a number of constraints making sure that we obtain a solution which fulfils our requirements.

$$\text{Min } \sum_{i=1}^n \sum_{j=i}^n b_{ij} \sum_{k=1}^{i-1} \sum_{l=i}^n (x_{kl} + x_{lk}) \quad (1)$$

Subject to:

$$\sum_{j=i}^n b_{ij} \leq 1 \quad i = 2, \dots, n \quad (2)$$

$$\sum_{i=1}^j b_{ij} \leq 1 \quad j = 1, \dots, n-1 \quad (3)$$

$$\sum_{j=1}^n b_{1j} = 1 \quad (4)$$

$$\sum_{j=1}^n b_{jn} = 1 \quad (5)$$

$$\sum_{i=1}^j b_{ij} = \sum_{k=j+1}^n b_{(j+1)k} \quad j = 1, \dots, n-1 \quad (6)$$

$$\sum_{i=1}^n \sum_{j=i}^n b_{ij} \leq A_{\max} \quad (7)$$

$$\sum_{i=1}^n \sum_{j=i}^n b_{ij} \geq A_{\min} \quad (8)$$

$$b_{ij}(d_{ij} - D_{\min}) \geq 0 \quad i = 1, \dots, n, \quad j = i, \dots, n \quad (9)$$

$$b_{ij}(D_{\max} - d_{ij}) \geq 0 \quad i = 1, \dots, n, \quad j = i, \dots, n \quad (10)$$

$$b_{ii} = 0 \quad i = 1, \dots, n, \quad (11)$$

Figure 4: Mathematical formulation of earthwork section division problem.

In the following we want to deduce the constraints appearing in our BP formulation (in Figure 4). As mentioned before, we do not want earthwork sections to overlap. In particular this means that at each position p_i , $i = 1, \dots, n$, at most one section can begin or end, which is expressed by inequalities (2) and (3). The special cases to consider, namely the first and last position, where an earthwork section has to begin and end, respectively, are captured in constraint (4) and (5). Since we want to cover the whole road project with our section division, it is necessary that a new section begins right after a section ends. Conversely, no section can start at position p_{i+1} without the preceding section

ending at position p_i , if we disallow gaps. These properties are guaranteed by condition (6) in our program, since by (2) and (3) both of the sums in (6) can only admit the values zero or one. Observe that these constraints also prevent overlapping sections. Hence, the aforementioned (in-)equalities along with the variable definition (12) are sufficient to describe the feasible section divisions.

Nevertheless, it may be useful to add several other constraints in order to avoid trivial solutions, such as the section division only consisting of one section. The addition of (7) and (8) with user defined integers A_{\max} and A_{\min} filters out all section divisions in which the number of resulting earthwork sections exceeds A_{\max} or is below A_{\min} . Let d_{ij} denote the actual distance between positions p_i and p_j . Then in a similar manner constraints (9) and (10) ensure that each earthwork section has a minimal length of D_{\min} and a maximal length of D_{\max} , where D_{\min} and D_{\max} are user defined values. Optionally, the addition of condition (11) makes sure that all earthwork sections include more than just a single cut or fill area. As stated before, the objective of our optimization problem is to find a section division with minimal intersectional earth movement. Therefore it is straightforward idea to define the objective function value for a feasible division by simply summing up all material flows between different earthwork sections. However, the material flow between two non-adjacent sections also influences all intermediate sections and therefore should be especially punished.

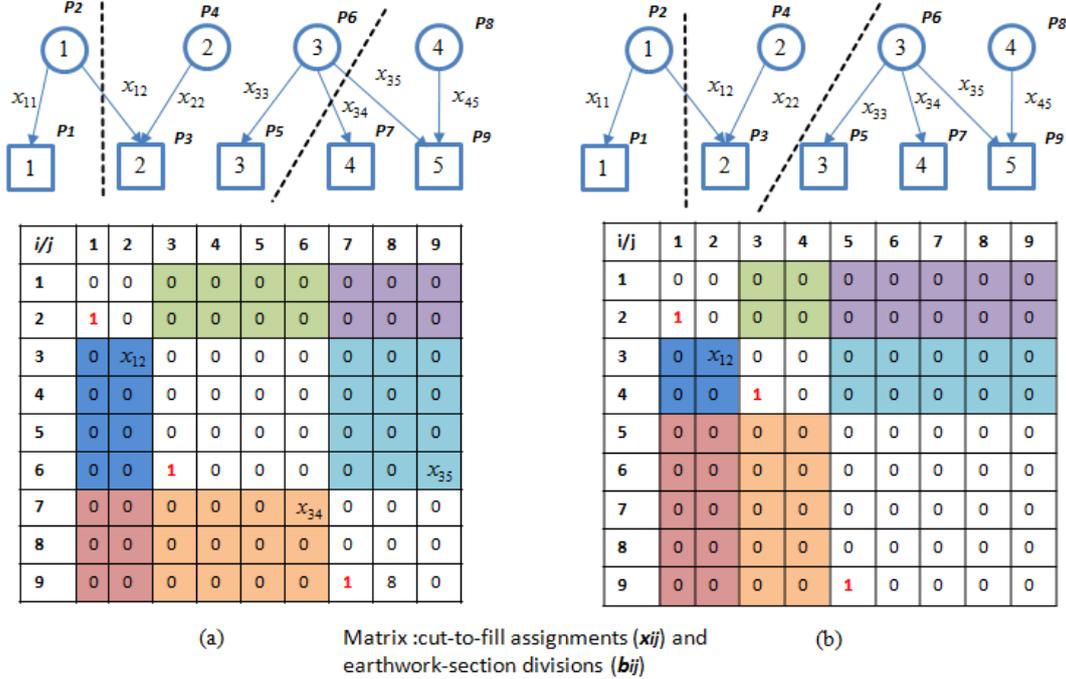


Figure 5. Cut-to-fill assignment matrix (x_{ij}) _{ij} obtained from the *earthwork allocation problem* and matrix (b_{ij}) _{ij} represent the earthwork section divisions. (a) Solution with three earthwork sections $ES_1 = \{p_1, p_2\}$, $ES_2 = \{p_3, p_4, p_5, p_6\}$ and $ES_3 = \{p_7, p_8, p_9\}$, represented by $b_{12} = b_{36} = b_{79} = 1$, with objective value $x_{12} + x_{34} + x_{35}$; (b) solution $b_{12} = b_{34} = b_{59} = 1$ with corresponding sections $ES_1 = \{p_1, p_2\}$, $ES_2 = \{p_3, p_4\}$ and $ES_3 = \{p_5, p_6, p_7, p_8, p_9\}$, having the better objective value x_{12} .

In our objective function (1), for each earthwork section in the division ($b_{ij} = 1$) we add the term $\sum_{k=1}^{i-1} \sum_{l=i}^n (x_{kl} + x_{lk})$, which expresses the sum of material flows passing the starting position p_i . By doing so, we also count the material to be transported beyond the ending position of a section, if

existent, since in a feasible division another section has to begin in the subsequent position. Consequently, a material flow x_{kl} is counted each time it crosses the border of a section. An example for feasible earthwork section divisions with different objective value is presented in Figure 5.

4 Real-world example

A large federal highway construction project has been planned to be constructed in Germany in the next year. The linear construction site which consists of 41 cut and fill areas is about 20 kilometers long. As we can see in the following figure, the cut and fill areas are distributed along the entire project construction site.

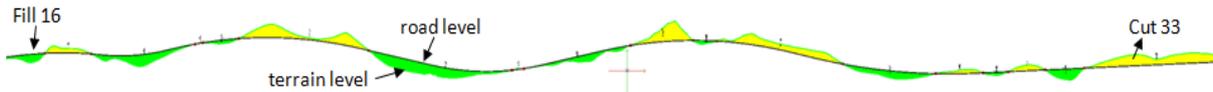


Figure 6. Part of vertical alignment of the road construction project.

The list of earthwork quantities corresponding to the cut and fill areas in Figure 6 are presented in Table 1 of Figure 7, as well as the optimal cut-to-fill assignments resulted from solving earthwork allocation problem.

Table 1. Part of cut and fill quantities			Table 2. Part of results of earthwork allocation problem.		
Position	Type	Volume (m3)	From	To	Volume(m3)
...
16	Fill	1470.5	Cut 7	Fill 16	845.2
17	Cut	965.1	Cut 7	Fill 18	502.8
18	Fill	2733.3	Cut 8	Fill 18	10.0
19	Cut	4.5	Cut 15	Fill 16	625.3
20	Fill	396.2	Cut 17	Fill 18	965.17
21	Cut	4197.9	Cut19	Fill 18	4.5
22	Fill	4999.1	Cut 21	Fill 18	1251.7
23	Cut	22.1	Cut 21	Fill 20	396.2
...

Figure 7. Part of earthwork quantities and result of earthwork allocation problem.

The large road project will be tendered to 3 different construction companies, and each construction section must have a length of 4 kilometers, at least. This earthwork section division problem can be formulated and solved using the binary linear program provided in this paper. In Figure 8, the optimal earthwork section division regarding user given parameters is illustrated.

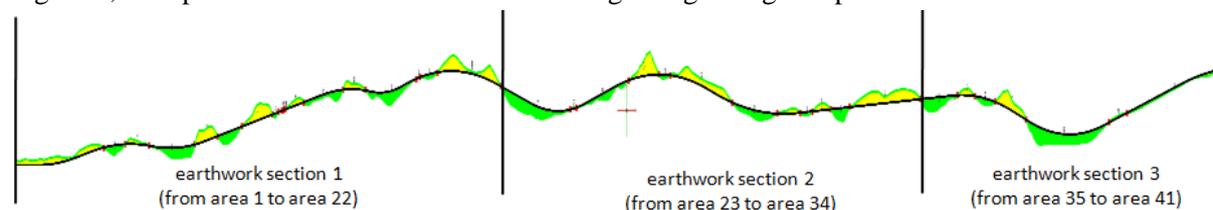


Figure 8. Optimal earthwork section divisions: 3 earthwork section, each section at least 4 kilometers long.

With the integration of the powerful open-source linear and mix-integer programming solver (GLKP v.4.3, 2009) into the earthwork assessment system *ForBAU Integrator* (Ji et al., 2009), the solutions of the two optimization problems can be found in acceptable running time, e.g. for dividing 41 earthwork areas, the solver computes the optimal solution within 3 seconds on a common machine.

5 Conclusion and future research

This paper introduces two major problems arising in optimizing earthwork processes: finding the most cost-efficient cut-to-fill-assignments (*earthwork allocation problem*) and dividing a large earthwork project into sections with minimal inter-sectional material flows (*earthwork section division problem*). This paper also presents the mathematical formulation and solution model of these two problems using (binary) linear programming technique. The introduced models and their solutions are applied in a real-world construction project, a highway construction site in Germany, to enhance the productivity in construction project.

In future research, we aim at solving two further optimization problems focusing on minimizing earth transport equipments and the project duration:

- With a given number of transporters, what is the minimal earthwork duration?
- To the given earthwork duration, what is the minimal number of transporters required to execute all transportation within the prescribed duration?

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